Chapter 5 - Day 2

The Product Rule

Let $f(x)$ and $g(x)$ be differentiable functions.

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

$$\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}(f(x))g(x) + f(x)\frac{d}{dx}(g(x))$$

Ex: Let $y=(2x+1)(x^2+2)$. Find y'

$$\begin{aligned} y' &= (2x+1)'(x^2+2) + (2x+1)(x^2+2)' \\ &= 2(x^2+2) + (2x+1)(2x) \\ &= 2x^2+4 + 4x^2+2x \\ &= 6x^2+2x+4 \end{aligned}$$

Ex: let $h(x) = x^2 + 4x + 1$, $g(2) = 5$, $g'(2) = 1$, and $F(x) = g(x)h(x)$.

Find $\left.\frac{dF}{dx}\right|_{x=2}$

* "the derivative of F with respect to x at x=2"

$$\left.\frac{dF}{dx}\right|_{x=2} = g'(2)\,h(2) + g(2)\,h'(2)$$

we need h(2) and h'(2)

$h(2) = 2^2 + 4(2) + 1 = 13$

$h'(x) = 2x + 4$

$h'(2) = 2(2) + 4 = 8$

So $\left.\frac{dF}{dx}\right|_{x=2} = g'(2)\,h(2) + g(2)\,h'(2)$

$= 1 \cdot 13 + 5 \cdot 8$

$= \boxed{53}$

The Quotient Rule

let $f(x)$ and $g(x)$ be differentiable functions.

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2}$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}(f(x))\,g(x) - f(x)\,\frac{d}{dx}(g(x))}{[g(x)]^2}$$

Ex: find $g'(x)$ for $g(x) = \dfrac{3x+2}{4-5x}$

$$g'(x) = \frac{(3x+2)'(4-5x) - (3x+2)(4-5x)'}{(4-5x)^2}$$

$$= \frac{3(4-5x) - (3x+2)(-5)}{(4-5x)^2}$$

$$= \frac{12 - 15x + 15x + 10}{(4-5x)^2} = \frac{22}{(4-5x)^2}$$

Ex: The equation of the tangent line to the graph of $g(x)$ at $x=9$ is given by

$$y = 21 + 2(x-9)$$

Find $g(9)$ and $g'(9)$.

$$y - 21 = 2(x-9)$$

point $(9,21)$ $\quad$ $g(9) = 21$

slope $= 2$ $\quad$ $g'(9) = 2$

Composite Functions

a function $h(x)$ is called a composite function of $f(x)$ followed by $g(x)$ if

$$h(x) = (g \circ f)(x) = g(f(x))$$

Ex: find functions $f(x)$ and $g(x)$ such that $h(x) = g(f(x))$.

a) $h(x) = (x^3 + 4x + 11)^7$

We have a polynomial & an exponent

$f(x) = x^3 + 4x + 11$

$g(x) = x^7$

b) $h(x) = \sqrt{x^2+3}$

there is a square root & a polynomial.

$f(x) = x^2+3$

$g(x) = \sqrt{x}$

The Chain Rule

$f(x)$ and $g(x)$ are functions, with $f(x)$ differentiable at x and $g(x)$ differentiable at the point $f(x)$.

$$(g(f(x)))' = g'(f(x))\,f'(x)$$

if $y = g(u)$ and $u = f(x)$ then $y = g(u) = g(f(x))$

and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Ex: let $K(x)=(1+3x^2)^3$. find $k'(x)$.

$$k'(x)=3(1+3x^2)^2\cdot(6x)$$
$$=3(1+6x^2+9x^4)(6x)$$
$$=18x+108x^3+162x^5$$

Ex: let $g(s)=(2s^3-3s^2+9)^5$. find $\frac{dg}{ds}$.

$$\frac{dg}{ds}=5(2s^3-3s^2+9)^4\cdot(6s^2-6s)$$
$$=(30s^2-30s)(2s^3-3s^2+9)^4$$

Ex: $f(t) = \sqrt{t^2+5}$ find $f'(t)$

$f(t) = (t^2+5)^{1/2}$

so

$f'(t) = \frac{1}{2}(t^2+5)^{-1/2}(2t)$

$= \dfrac{t}{(t^2+5)^{1/2}}$

$= \dfrac{t}{\sqrt{t^2+5}}$